On Hájek's Fuzzy Quantifiers "Probably" and "Many"

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Łukasiewicz logic Ł

Connectives: implication $\to$ and 'falsum' $\perp$ (we set $\neg\varphi = \varphi \to \perp$)

(Standard) semantics: evaluation is a mapping $e\colon \mathrm{FOR} \to [0,1]$ st:

$$e(\perp) = 0 \qquad\qquad e(\varphi \to \psi) = \min\{1, 1 - e(\varphi) + e(\psi)\}$$

Axiomatic system: deduction rule is *Modus Ponens* (from φ and $\varphi \to \psi$ infer ψ); axioms are:

$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$$

Completeness: Thm(Ł) = Taut(Ł)

More on Łukasiewicz logic

Troubles with connectives

$$\varphi \wedge \psi \quad \equiv_{\text{Bool}} \quad \neg(\varphi \to \neg\psi) \quad \equiv_{\text{Bool}} \quad \neg((\psi \to \varphi) \to \neg\psi)$$

$$\neg(\tfrac{1}{2} \to \neg\tfrac{1}{2}) = 0 \qquad \neg((\tfrac{1}{2} \to \tfrac{1}{2}) \to \neg\tfrac{1}{2}) = \tfrac{1}{2}$$

Thus we define:

$$\varphi \wedge \psi = \neg((\psi \to \varphi) \to \neg\psi) \qquad e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$$

$$\varphi \,\&\, \psi = \neg(\varphi \to \neg\psi) \qquad e(\varphi \,\&\, \psi) = \max(0, e(\varphi) + e(\psi) - 1)$$

Funny observation:

$(\varphi \wedge \neg\varphi) \to \bot$	IS NOT	provable in Ł
$(\varphi \,\&\, \neg\varphi) \to \bot$	IS	provable in Ł

Two useful connectives:

$$\varphi \oplus \psi = \neg\varphi \to \psi \qquad \min(1, e(\varphi) + e(\psi))$$

$$\varphi \ominus \psi = \neg(\varphi \to \psi) \qquad \max(0, e(\varphi) - e(\psi))$$

On two conjunctions

& is not idempotent! Girard example:

A) If I have one dollar, I can buy a pack of Marlboros $D \to M$

B) If I have one dollar, I can buy a pack of Camels $D \to C$

Therefore: $D \to M \wedge C$ i.e.,

C) If I have one dollar, I can buy a pack of Ms and pack of Cs

BETTER: $D \,\&\, D \to M \,\&\, C$ i.e.,

C′) If I have one dollar and I have one dollar,
I can buy a pack of Ms and pack of Cs

On two conjunctions (cont.)

Consider three glasses of beer: 0.3L, 0.5L, and 1L.

Consider predicates $P_a(x)$: 'Petr can drink x in a minutes'

	P_1	P_2	P_3
0.3L	1	1	1
0.5L	0	1	1
1L	0	0	1

- There is a beer Petr can drink in one minute — TRUE
- Petr can drink any of the beers in two minutes — FALSE
- Petr can drink *any of the* beers in three minutes — TRUE
- Petr can drink *all the* beers in three minutes — FALSE

$$(\forall x)\varphi \to \varphi(a) \wedge \varphi(b) \qquad \text{but not} \qquad (\forall x)\varphi \to \varphi(a) \,\&\, \varphi(b)$$

$$\varphi \wedge \psi \to \chi \qquad \text{is equivalent to} \qquad (\varphi \to \chi) \vee (\psi \to \chi)$$

On transitivity

We define a ‘fuzzy indistinguishability’ relation

$$Exy = \max\{0, 1 - |x - y|\}$$

Then in Łukasiewicz logics holds:

$$||Exy \,\&\, Eyz \to Exz|| = 1$$

Re define other fuzzy relation (assume that $0 \leq a \leq 1$):

$$E_a xy = \min(1, \max(0, 1 + a - |x - y|))$$

Note that if $|x - y| \leq a$ then $E_a xy = 1$

Then in Łukasiewicz logics holds:

$$||(\forall xyz)(Exy \,\&\, Eyz \to Exz)|| = 1 - a$$

On generalized quantifiers

Note: (generalized) quantifiers are functions from sets of individuals to $\{0, 1\}$

Thus: generalized quantifiers are special **unary predicates**

Thus our proposal is obvious:

fuzzy generalized quantifiers are functions from sets of individuals to [0,1]

Note: if most participant are vegetarians, most of the food at the banquet is vegetarian

Probability inside Łukasiewicz logic: language

The language of $\mathbf{FP}(\text{Ł})$ has a non-empty set V of the crisp (two-valued) propositional variables. It has three kinds of formulas:

- **NON-MODAL:** The formulas built from the propositional variables in the usual way, using crisp connectives $\wedge$ and $\neg$ i.e., a classical formulas

- **ATOMIC MODAL:** The formulas built from the non-modal formulas by using new fuzzy modality P i.e., formulas $P\varphi$, where φ is the non-modal formula,

- **EXTENDED MODAL:** The formulas built from the atomic modal formulas in the usual way, using connectives of the Łukasiewicz logic: $\neg_{\text{Ł}}$, $\rightarrow_{\text{Ł}}$.

Probability inside Łukasiewicz logic: semantics

The models of $\mathbf{FP}(\text{Ł})$ are probability Kripke structure $\mathbf{K} = \langle W, e, \mu \rangle$ where:

- W is a non empty set of possible worlds,
- $e\colon W \times \mathsf{VAR} \to \{0, 1\}$ is a crisp evaluation of the propositional variables in each world
- $\mu\colon 2^W \to [0, 1]$ is a finitely additive probability measure st. for each variable p, the set $\{w \mid e(w, p) = 1\}$ is measurable.

Probability inside Łukasiewicz logic: definition of truth

Let $\mathbf{K} = \langle W, e, \mu \rangle$ be a probability Kripke structure. The evaluation e can be extended to the formulas of the $\mathbf{FP}(\text{Ł})$:

- **NON-MODAL:** an usual extension of the evaluation of the propositional variables to the non-modal formulas.

- **ATOMIC MODAL:** $e(\widehat{w}, P\varphi) = \mu\{w \mid e(w, \varphi) = 1\}$

- **EXTENDED MODAL:** also an usual extension of the evaluation of the atomic modal formulas to the modal formulas

Probability inside Łukasiewicz logic: axiomatic system

(FP0) the axioms of the Łukasiewicz logic

(BOOL) $\varphi \vee \neg\varphi$ for non-modal φ

(FP1) $P(\varphi \to \psi) \to_{Ł} (P\varphi \to_{Ł} P\psi)$

(FP2) $\neg_{Ł} P(\varphi) \to_{Ł} P(\neg\varphi)$

(FP3) $P(\varphi \vee \psi) \to_{Ł} ((P\psi \oplus (P\varphi \ominus P(\varphi \wedge \psi)))$

The deduction rules are modus ponens and the necessitation of P: from φ infer $P\varphi$ (for φ being a non-modal formula)

Probability inside Łukasiewicz logic: completeness

Completeness: Let Ψ be a modal formula and **T** a finite modal theory over $\mathbf{FP}(\text{Ł})$. Then $\mathbf{T} \vdash \Psi$ iff $e_{\mathbf{K}}(\Psi) = 1$ for each probability model $\mathbf{K}$ of the theory **T**.

Particular cases and modifications:

Quantifier 'many' (in KF with n worlds)

$$e(\widehat{w}, M\varphi) = \frac{1}{n} \sum_{w \in W} e(w, \varphi)$$

Modification of definition of semantics

- $e\colon W \times \mathsf{VAR} \to [0, 1]$
- $\mu_w\colon [0, 1]^W \to [0, 1]$ is any function

Thank you for you attention

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(and sorry for the examples)